"AXIAL ANOMALY" WS 12/13.

Plan:

1. Introduction. Definition of the anomaly.
2. Derivation of the axial anomaly.
3. Applications.
4. Physical origins of the anomaly.
5. A modern perspective (if we have time).

Literature:
1) K. Fujikawa, Phys. Rev. D21, 2848
2) V.I. Zakharov, Phys. Atom. Nucl. 73, 895.
3) Peskin-Schröder.

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① Def. Anomaly is a violation of the Ehrenfest theorem.

We'll consider one specific case of axial current non conservation. Basic objects:

$j^\mu = \bar\Psi \gamma^\mu \Psi$ — vector (electric) currents.

$j_5^\mu = \bar\Psi \gamma^\mu \gamma^5 \Psi$ — axial singlet currents.

$j_5^{\mu a} = \bar\Psi \gamma^\mu \gamma^5 T^a \Psi$ — axial non-singlet currents

T^a — generators of $SU(N_f)$.

Physical meaning of $\bar\Psi \gamma^5 \Psi$ is the difference in densities of left- and right-handed quarks

(recall that $\gamma_5 = \begin{pmatrix} -\mathbb{1}_{2\times2} & \\ & \mathbb{1}_{2\times2} \end{pmatrix}$.)

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then we can think of j_5^μ as a propagating difference in densities.

At the classical level

$$\partial_\mu j^\mu = \partial_\mu j_5^\mu = \partial_\mu j_5^{\mu a} = 0$$

(we'll show it later), but at the quantum level, keeping $\partial_\mu j^\mu = 0$, we have for massless fermions and Abelian fields

$$\partial_\mu j_5^\mu = \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \qquad \tilde{F}_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

[Adler, Bell, Jackiw '69].

Another example is the conformal anomaly,

$\langle T^\mu{}_\mu \rangle \neq 0$, while classically $T^\mu{}_\mu = 0$.

(will be presented at later lectures)

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(2) Let's derive the ABJ anomaly.

we start from $N_f = 1$ fermions coupled to $SU(N)$ Yang-Mills field,

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) + \bar{\Psi}(i \not{D} - m)\Psi$$

$$\not{D} \equiv \gamma^\alpha D_\alpha \equiv \gamma^\alpha (\partial_\alpha + A_\alpha)$$

Wick rotation:

$$x^0 \to -i x^4$$
$$A_0 \to i A_4$$
$$\not{D} \to i\gamma^0 D_4 + \gamma^k D_k = \gamma^4 D_4 + \gamma^k D_k$$
$$g_{\mu\nu} \to \mathrm{diag}(-1,-1,-1,-1)$$

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It's important that we have then a Hermitian Dirac operator and anti-Hermitian $\gamma^{1,2,3,4}$

$\gamma_E^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$

Additional (standard) conventions:

$i\mathcal{A}_\mu = g\,A_\mu^a T^a$, $\quad T^a$ - generators of $SU(N)$

$[T^a, T^b] = i f^{abc} T^c$,

$\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$,

$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu \mathcal{A}_\nu]$.

To define the functional integral, we have an option to consider fermionic fields in a basis of eigenfunctions of $\not{D}$:

$$\psi(x) = \sum_n a_n \varphi_n(x)$$

$$\bar{\psi}(x) = \sum_n \varphi_n^+(x)\, \bar{b}_n$$

Grassmann numbers,

where $\not{D}\,\varphi_n(x) = \lambda_n \varphi_n(x)$ and

$$\int d^4x\; \varphi_n^+(x)\,\varphi_m(x) = \delta_{n,m}.$$

Path integral: $\int d\bar{\psi}\, d\psi\; e^{-S_E} =$

$$= \int \prod_{n=1}^{\infty} d\bar{b}_n\, da_n \times \exp\left\{ -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] - \sum_{n,k=1}^{\infty} \bar{b}_n \varphi_n^+(x)(i\not{D}-m)\varphi_k(x)\, a_k \right\}$$

$$= e^{-S_{YM}} \times \det(i\not{D}-m).$$

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Let's perform the chiral rotation

$$\begin{cases} \Psi(x) \to e^{i\alpha(x)\gamma_5}\Psi(x) \\ \bar\Psi(x) \to \bar\Psi(x)\, e^{i\alpha(x)\gamma_5} \end{cases}$$

then $\quad \mathcal{L} \to \mathcal{L} - \partial_\mu \alpha(x)\, \bar\Psi \gamma^\mu \gamma^5 \Psi(x) - 2mi\,\alpha(x)\,\bar\Psi\gamma^5\Psi.$

So, in the chiral limit we would expect

$$\partial_\mu\left(\bar\Psi\gamma^\mu\gamma^5\Psi\right) = 0, \quad \text{where is the anomaly?}$$

Actually we have to check, how the integration measure is transformed!

$$a_m \to \sum_n \int d^4x\, \varphi_m^+(x)\, e^{i\alpha(x)\gamma_5}\varphi_n(x)\, a_n \equiv a_m' \equiv \sum_n C_{mn} a_n$$

Indeed, $\displaystyle\sum_m a_m'\varphi_m(x) = \sum_{m,n}\int d^4x'\, \varphi_{\underline{\underline{m}}}^+(x')\, e^{i\alpha(x')\gamma_5}\varphi_n(x')\, a_n\, \varphi_{\underline{\underline{m}}}(x)$

$$= \sum_n e^{i\alpha(x)\gamma_5}\varphi_n(x)\, a_n$$

So, the Jacobian arises:

$$\prod_{m=1}^{\infty} d\bar b_m\, d a_m \to \left[\det C_{mn}\right]^{-2} \prod_{n=1}^{\infty} d\bar b_n\, d a_n .$$

for infinitesimal $\alpha(x)$ we obtain

$$\det C_{mn} = \det\left[\delta_{m,n} + i\int \alpha(x)\,\varphi_m(x)^+\gamma_5\varphi_n(x)\,dx\right] =$$

$$= \exp\left\{ i\sum_n \int dx\, \alpha(x)\, \varphi_n(x)^+\gamma_5\,\varphi_n(x)\right\} \equiv$$

$$\equiv \exp\left\{ i\int dx\ \alpha(x)\, \mathcal{A}(x)\right\},$$

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where we used the identity $\underline{\det = \exp \mathrm{Tr} \ln}$

Now we have to calculate $\mathcal{A}(x)$ to define the Jacobian $\exp\{-2i \int dx\, \alpha(x)\, \mathcal{A}(x)\}$.

In fact, $\mathcal{A}(x)$ is ill-defined because naively it is simply

$$\mathrm{Tr}\, \gamma^5 \cdot \delta(x-x) = 0 \times \infty .$$

To resolve this uncertainty, we can introduce a regularization, a cut-off in the fermionic spectrum: $|\lambda_n| \lesssim M$

$$\mathcal{A}(x) = \lim_{M\to\infty} \Big(\sum_n \varphi_n(x)^+ \gamma_5 e^{-(\lambda_n/M)^2} \varphi_n(x) \Big) =$$

"projector"

$$= \lim_{M\to\infty} \Big(\sum_n \varphi_n(x)^+ \gamma_5 e^{-(\not{D}/M)^2} \varphi_n(x) \Big) =$$

$$= \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 e^{ikx} e^{-(\not{D}/M)^2} e^{ikx} =$$

$$= \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 \times$$

$$\times \exp\Big\{ -\frac{1}{2M^2} \Big[2 (ik_\mu + D_\mu(x))^2 + \frac{1}{2} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}(x) \Big] \Big\}$$

(here we made a shift of the derivative and used $\not{D}^2 = D^2 + \frac{1}{4} [\gamma^\mu \gamma^\nu] F_{\mu\nu}$)

$$= \lim_{M\to\infty} \mathrm{Tr}\, \gamma^5 \left(\frac{1}{2}[\gamma^\mu,\gamma^\nu] F_{\mu\nu}\right)^2 \cdot \left(\frac{1}{2M^2}\right)^2 \frac{1}{2!} \int \frac{d^4k}{(2\pi)^4}\, e^{-\frac{k^\mu k_\mu}{M^2}}$$

$$= \frac{1}{2}\left(-\frac{1}{8\pi^2}\right) \mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \text{where} \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

So, finally, $\partial_\mu j_5^\mu = -2mi\,\bar{\psi}\gamma^5\psi + \frac{i}{8\pi^2}\mathrm{Tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$.

we can also consider Abelian fields and $m \to 0$. The general expression will then be

$$\boxed{\partial_\mu j_5^\mu = -\frac{C_\gamma}{4} F^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{C_g}{4} F^{\mu\nu\, a}\tilde{F}_{\mu\nu}^{a}}$$

$C_\gamma \equiv \frac{e^2}{2\pi^2} N_c$, $\quad C_g = \frac{g^2}{4\pi^2}$

(in Minkowski metric)

e.m. fields

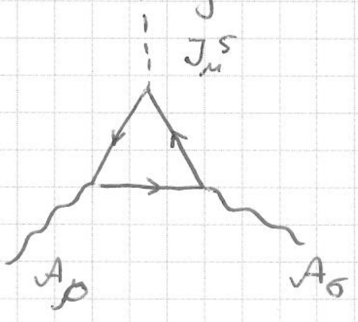
gluonic fields.

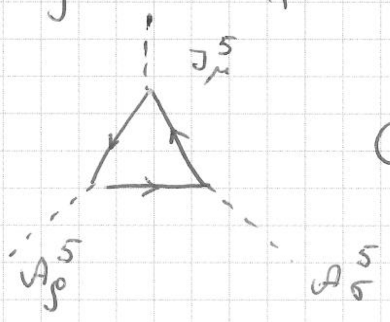
<u>Remarks</u>:

1) One can force j_5^μ to be conserved, but then j^μ will be anomalous, so we loose gauge-invariance (see Peskin-Schröder).

2) the Feynman diagrams for the anomaly:

C_γ





$C_\gamma/3$.

one can shift the anomaly from one current to another (see Weinberg v.2).

3) We can keep M finite [Andrianov '84].

4) In QCD we have to keep it finite, since there are nonperturbative scales $\gg \Lambda_{QCD}$ [1208.0012].

5) At $N_c \to \infty$ the non-Abelian anomaly dies out.

6) the expression for anomaly is valid in all orders of the perturbation theory [Adler and Bardeen '69].

7) there are also other ways to derive the axial anomaly.

7.1) Feynman diagrams (see e.g. Peskin-Schröder).

7.2) From IR divergency of the dispersive integral [Dolgov, Zakharov '71].

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③ Applications

3.1) $\pi^0 \to \gamma\gamma$ decay rate.

$$j^\mu_{5\,(\Delta T=1)} = \frac{1}{\sqrt{2}} \left(\bar{u} \gamma^\mu \gamma^5 u - \bar{d} \gamma^\mu \gamma^5 d \right),$$

then from the triangle diagrams

(*) $$\langle \gamma\gamma | j_5^\mu | 0 \rangle = \frac{i\alpha}{4\sqrt{2}\pi} \left(Q_u^2 - Q_d^2 \right) N_c \frac{q^\mu}{q^2} F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$

pole!

In the limit $m_{u,d} \to 0$ there is only one (neutral) massless state in the hadronic spectrum, namely π^0.

(**) $\langle \gamma\gamma | j_5^\mu | 0 \rangle_{\text{pole}} = f_\pi A(\pi^0 \to \gamma\gamma) \frac{q^\mu}{q^2} F\tilde{F}$, where

$\langle 0 | j_5^\mu | \pi^0 \rangle = q^\mu f_\pi$.

Comparing (*) with (**) we get

$$A(\pi^0 \to \gamma\gamma) = \frac{\alpha}{4\sqrt{2}\pi f_\pi} N_c \left(Q_u^2 - Q_d^2\right).$$

[’t Hooft ’75].

3.2) Renormalizability of the theory of weak int.

If a current is a source of a gauge boson, then it should be conserved.

$$j_{5\mu}^Z = \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d + \bar{\nu}\gamma_\mu\gamma_5\nu - \bar{e}\gamma_\mu\gamma_5 e\right).$$

each of the terms induces an anomaly, but together they must cancel each other, so

$$\boxed{N_c\left(Q_u^2 - Q_d^2\right) - Q_e^2 = 0}$$

For $N_c = 3$ we obtain non-integer charges of quarks, e.g., as we expect, $Q_u = 2/3$, $Q_d = -1/3$.

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3.3) Again, in the limit $m_u = m_d = m_s \to 0$ we have nine classically conserved currents

$$\partial_\mu (\bar{q}_i \gamma^\mu \gamma^5 q_k)_{class} = 0, \qquad i,k = u,d,s.$$

So, we could expect nine Goldstone bosons. In reality we have one of them massive $m_{\eta'} = 958$ MeV. The contradiction is resolved by the following anomaly

(***) $$\partial_\mu (\bar{u}\gamma^\mu\gamma^5 u + \bar{d}\gamma^\mu\gamma^5 d + \bar{s}\gamma^\mu\gamma^5 s) = 3\frac{\alpha_s}{4\pi} F_{\mu\nu}^a \tilde{F}^{\mu\nu a}$$

[Fritzsch, Gell-Mann, Leutwyler '73, 't Hooft '76].

<u>Sketch</u>: $\langle 0 | J^\mu | \eta' \rangle \sim p^\mu \cdot f_{\eta'}$

$$\langle 0 | \partial_\mu J^\mu | \eta' \rangle \sim m_{\eta'}^2 \cdot f_\pi \qquad (f_\pi \sim f_{\eta'})$$

$\partial_\mu J^\mu = 0 \iff m_{\eta'}^2 = 0$, but it's not zero.

More precisely:

$$\frac{f_\pi^2}{6}\left(m_{\eta'}^2 + 2m_K^2 - m_\eta^2\right) = \chi,$$

where

$$\chi \equiv \lim_{V\to\infty} \frac{1}{V} \int d^4x \langle F_{\mu\nu}^a \tilde{F}^{\mu\nu a}(x)\, F_{\alpha\beta}^a \tilde{F}^{\alpha\beta a}(0) \rangle$$

is the topological susceptibility.
[Witten, Veneziano '79].

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④ Physical origin of the anomaly.

Let's define a "new" axial current

$$\tilde{j}_\mu^5 = j_\mu^5 - K_\mu, \quad \text{with} \quad K_\mu = \frac{\alpha}{4\pi}\varepsilon_{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma},$$

so $\quad \partial_\mu \tilde{j}^\mu_5 = 0 \qquad (m_e \to 0)$.

the charge $K_0 \sim \int d^3x \, \epsilon_{0ikl} A_i F_{kl}$ is gauge inv.!
also the chirality change

$$\Delta Q_A = \text{const} \cdot \int_{-T}^{+T} dt\, d^3x \; \underset{\partial_\mu K^\mu}{\underset{\shortparallel}{F\tilde{F}(x)}} = 0 \qquad (\text{tot. derivative})$$

In nonabelian theories (like QCD) it is not true, because of classical topological solutions.

- For instance, instanton

$$(G^2(x))_{inst.} = (G\tilde{G}(x))_{inst.} \sim \frac{1}{g^2} \frac{\rho^4}{(x^2+\rho^2)^4}$$

inst. radius.

$$S_I \sim \int d^4x \, (G^2)_{inst.} = \int d^4x \, (\tilde{G} G)_{inst} = \frac{8\pi^2}{g^2}$$

we see that it induces, via (***), a change of the axial charge

$$\Delta Q_A = 2 N_f.$$

Moreover, there is a solution for $\lambda = 0$ quarks

$$|q_{\lambda=0}(x)|^2 \sim \frac{\rho^4}{(\rho^2+x^2)^3}$$ localized in the same region of space as the instanton.

- One more interesting fact about zero-modes:

$$n_R^{(0)} - n_L^{(0)} = Q_A \qquad [\text{Atiyah-Singer theorem}].$$

- What part of the fermionic spectrum is responsible for the anomaly?

$\not{D}\Psi_\lambda = \lambda\Psi_\lambda$ as before,

$$\rho_5 \sim \langle \bar{\Psi}\gamma^5\Psi\rangle = i\sum_{\lambda\neq 0}\frac{\Psi_\lambda^+\gamma^5\Psi_\lambda}{\lambda - im} =$$

$$= i\sum_{\lambda>0}\frac{\Psi_\lambda\gamma^5\Psi_\lambda}{\lambda - im} + i\sum_{\lambda<0}\frac{\Psi_\lambda^+\gamma^5\gamma^5\gamma^5\Psi_\lambda}{-\lambda - im} =$$

$$= -2m\sum_{\lambda>0}\frac{\Psi_\lambda^+\gamma^5\Psi_\lambda}{\lambda^2+m^2} = -\int_0^\infty d\lambda\,\nu(\lambda)\frac{2m}{\lambda^2+m^2}\Psi_\lambda\gamma^5\Psi_\lambda,$$

here $\nu(\lambda)$ is the spectral density and we drop zero modes.

Then, using $\lim_{m\to 0}\frac{m}{\lambda^2+m^2} = \pi\delta(\lambda)$ we obtain

$$\lim_{m\to 0}\rho_5 = -2\pi\int_0^\infty d\lambda\,\nu(\lambda)\,\delta(\lambda)\,\Psi_\lambda^+\gamma^5\Psi_\lambda =$$

$$= -\pi\lim_{\lambda\to 0}\nu(\lambda)\,\Psi_\lambda^+\gamma^5\Psi_\lambda.$$

So, only the IR part of the fermionic spectrum is important! Exactly those part, which is localized around the classical gluonic solutions.

Also important that

$\nu(0) \sim \langle\text{chiral condensate}\rangle$.