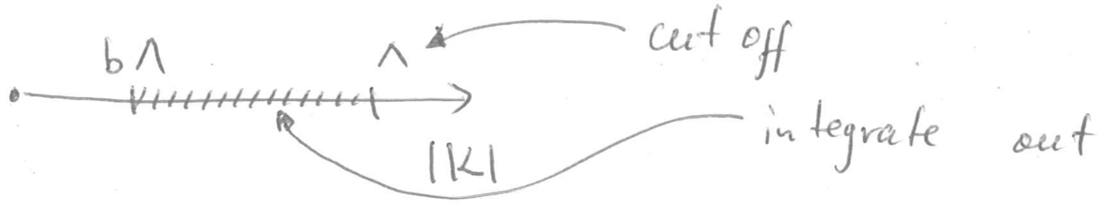
# Critical exponents: QFT methods for Statistical Phys.

**Literature:** 1. Peskin - Schröder "Introduction to QFT".
2. Previous lectures (e.g. by Misha).

(1) Purpose of the RG: it tells us how to go down to the low energies. What happens with operators (relevant, irrelevant, marginal).

Scalar theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$.

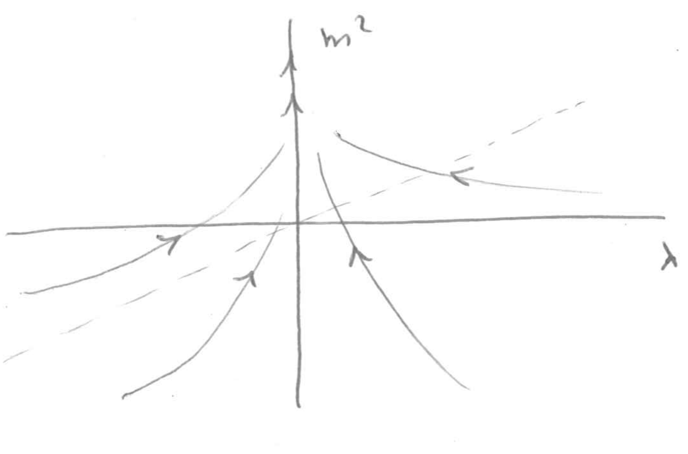



effects in $m^2 \to m^2 b^{-2}$

$\lambda \to \lambda b^{d-4}$

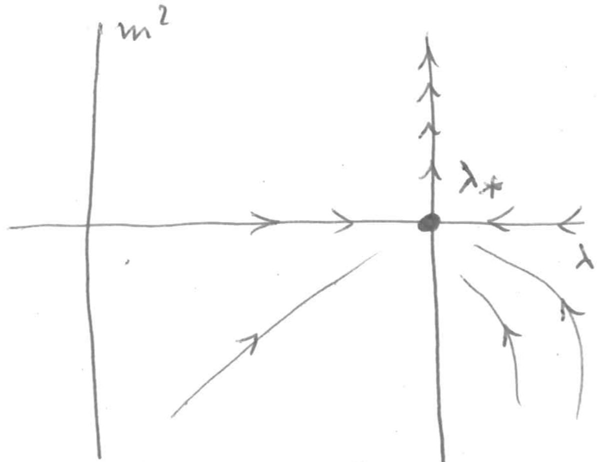
Multiplication by "b" can be considered as a trajectory in the configuration space:

$d \geq 4$




IR free theory

$d < 4$



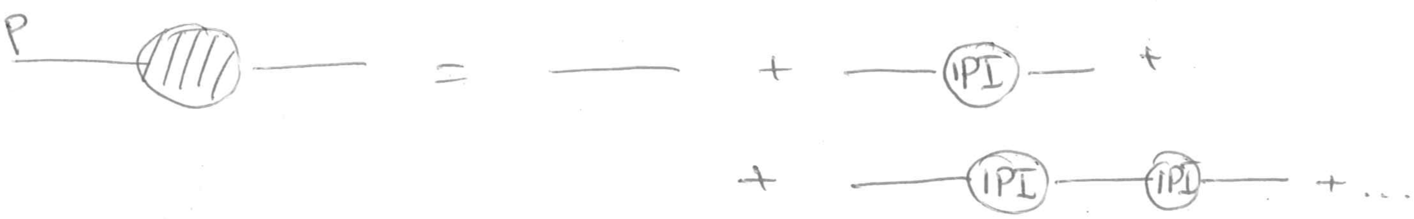

IR fixed point (Wilson-Fisher)

<u>Renormalizability</u>: if we send $\Lambda \to \infty$, then the RG flow still converges to some point in configuration space (one needs also adjustment for the bare parameters...).

This example is ill, since the mass grows. So, the Higgs mass grows until it reaches the cutoff. Typical disaster of scalar theories, because the mass is renormalized in an additive manner, not like for the fermions in QED.

So, $\phi^4$ is bad in QFT, but o.k. for cond. mat., where we know the natural $\Lambda \sim \frac{1}{a}$, where $a$ is the interatomic distance. (Close to $T_c$ we get a field theory, because $\xi \to \infty$ and we can forget about $a$ (i.e. $\Lambda$) and, hence, the universality class).

2) How do we fix the physical mass? It can be shifted:



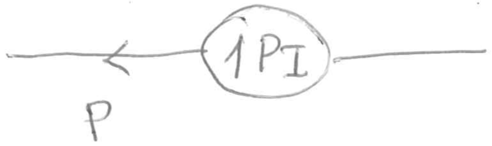
$$\underset{\text{scalar th.}}{=} \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}\left(-i\Pi(p^2)\right)\frac{i}{p^2-m_0^2} +$$

$$+ \frac{i}{p^2-m_0^2}\left(-i\Pi(p^2)\right)\frac{i}{p^2-m_0^2}\left(-i\Pi(p^2)\right)\frac{i}{p^2-m_0^2} + \dots$$

$$= \frac{i}{p^2-m_0^2-\Pi}$$

mass shift.

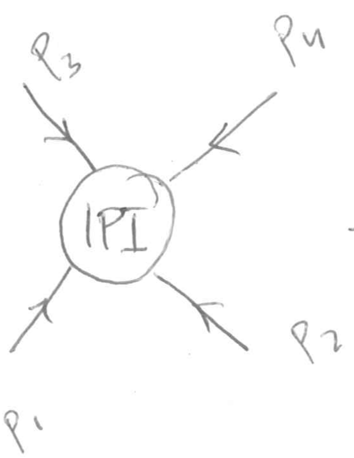
the way of fixing: renormalization conditions:




$= 0 \quad @ \; p^2 = m^2$

$$\frac{d}{dp^2}\Big($$ 

$$\Big) = 0 \quad @ \; p^2 = m^2$$




$$= -i\lambda \quad @ \quad \begin{aligned}(p_1+p_2)^2 &= 4m^2 \\ (p_1+p_4)^2 = (p_1+p_3)^2 &= 0\end{aligned}$$

this defines you physical (experimental) $m^2$ and $\lambda$.

what if $m=0$? then we're in troubles, because usually $\Pi \sim (\dots)\log(\Lambda/m)$.

the way out is to replace the conditions by ones at $p^2 = -M^2$, $(p_1+p_2)^2 = (..)^2 = (..)^2 = -M^2$,

where $M$ = "renormalization scale".

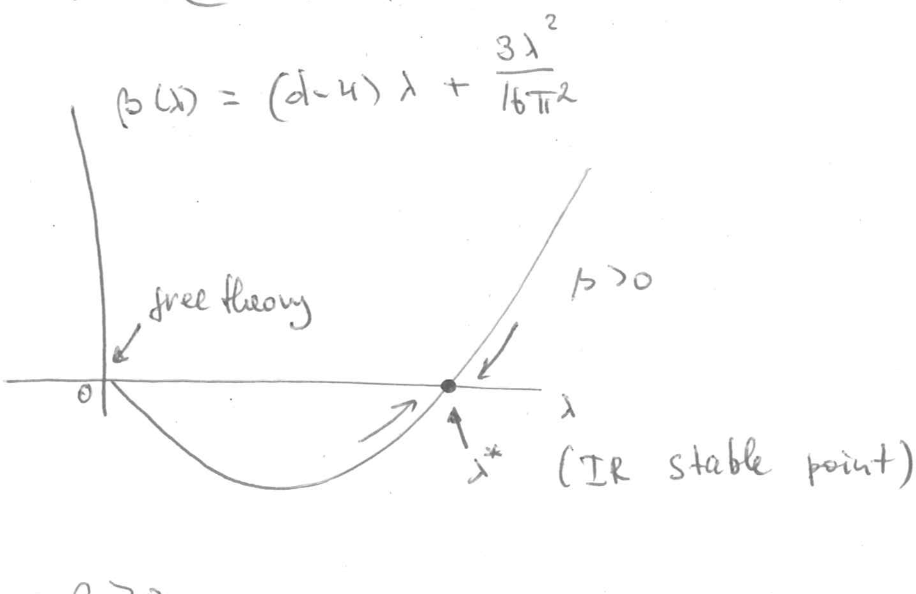
$M$ will enter everywhere. To make the changes consistent we introduce the Callan-Symanzin eqn:

$$\left[ M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + n\gamma \right] G^n(x_1 \dots x_n; M, \lambda) = 0$$

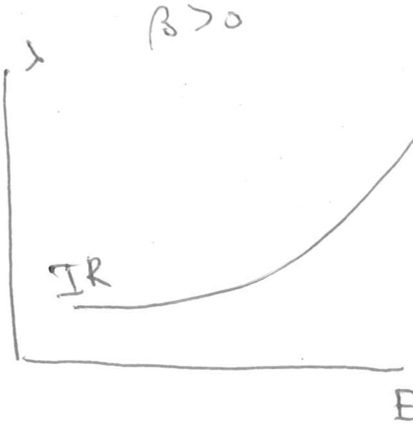
$\beta = \beta(\lambda)$ - shifts in the coupling constant.

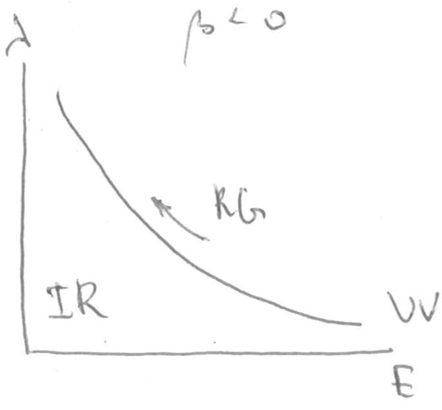
$\gamma = \gamma(\lambda)$ - shifts in the field strength.

Example: $\phi^4$ @ $d<4$




Example



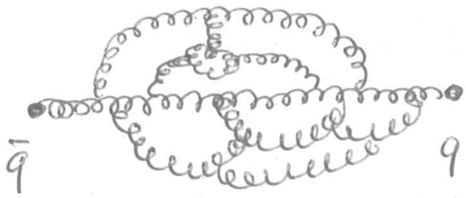
line QED


Coulomb's force


vacuum polarization

line QCD


Coulomb's force




flux tube

<u>Example of solution</u> (single scalar)

$$G^{(2)}(p) = \frac{i}{p^2}\, g\left(-p^2/M^2\right) \qquad \text{Ansatz}$$

then $P \equiv \sqrt{-p^2}$ and we can replace

$$\left(M\frac{\partial}{\partial M}\right) \text{ by } \left(-P\frac{\partial}{\partial P} - 2\right), \text{ so}$$

$$\left[P\frac{\partial}{\partial P} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2 - 2\gamma(\lambda)\right] G^{(2)}(P) = 0.$$

the solution is then

$$G^{(2)}(P,\lambda) = \tilde{G}\left(\bar{\lambda}(P;\lambda)\right)\cdot \exp\left\{-\int_{P'=M}^{P'=P} d\log\frac{P'}{M}\, \times\right.$$

$$\left. \times\, 2\left(1-\gamma\left(\bar{\lambda}(P';\lambda)\right)\right)\right\}, \quad \text{where } \bar{\lambda} \text{ solves}$$

$$\frac{d}{d\log(P/M)}\bar{\lambda}(P;\lambda) = \beta(\bar{\lambda}); \qquad \bar{\lambda}(M,\lambda) = \lambda.$$

the last one is the renormalization group eqn. with running coupling $\bar{\lambda}$.

③ Let's perturb the $\phi^4$ theory by additional operators,

$$L = L_{\text{old}} + \rho_i M^{4-d_i} \cdot \mathcal{O}_M^i(x),$$

here $\rho_i$ are dimensionless and $d_i$ is $\dim[\mathcal{O}_M^i]$.

then $$\left[ M\frac{\partial}{\partial M} + \beta \frac{\partial}{\partial \lambda} + n\gamma + \sum [\gamma_i(\lambda) + d_i - 4] \rho_i \frac{\partial}{\partial \rho_i} \right] G^{(n)} = 0$$

$\gamma_i(\lambda)$ is the anomalous dimension of $\mathcal{O}_i$.

what does it mean? Additional mass dimension due to the quantum corrections.

Example:

$$\mathcal{O}(x)\,\mathcal{O}(y) = \frac{1}{|x-y|^{2\Delta_0}} \left(1 - g^2 A \log(\Lambda|x-y|) + \dots\right) =$$

(series by powers of $g$)

$$= \frac{1}{|x-y|^{2\Delta}}, \text{ where now } \Delta = \Delta_0 + \underbrace{g^2 A + \dots}_{\text{anomalous dimension.}}$$

---

we can absorb : $\beta_i = (d_i - 4 + \gamma_i)\rho_i$

then the RG eqn. is not only one but with a set

$$\frac{d}{d\log(P/M)}\bar{\rho}_i = \beta_i(\bar{\rho}_i, \bar{\lambda})$$

if all $\lambda$ and $\beta_i$ are small, then

$$\frac{d}{d\log(P/M)}\bar{\rho}_i = [d_i - 4 + \dots]\,\bar{\rho}_i \quad \text{and} \quad \bar{\rho}_i = \rho_i\left(\frac{P}{M}\right)^{d_i-4}$$

Operators with $d_i > 4$ are irrelevant (i.e. @ $P \to 0$),
but $\varphi^2$ is relevant (mass operator)

## ④ Critical exponents.

Let's consider a magnet with spin field $S(x)$.
then $M \equiv \int dx \langle S(x) \rangle$ - magnetization

$$G(x) = \langle S(x)\, S(0) \rangle \sim \exp\left(-|x|/\xi\right) \quad \text{at } |x| \to \infty.$$

$\xi$ - correlation length.

Approach to the critical point is measured by $t \equiv \dfrac{T - T_c}{T_c}$.

if we had 2nd order phase transition, then $\xi \to \infty$.

$$\xi \sim |t|^{-\nu}, \qquad G(x) \sim \frac{1}{|x|^{d-2+\eta}}$$

Euclid. space dim.

$$C_H \sim |t|^{\alpha}, \qquad M \sim H^{1/\delta} \text{ @ } t=0.$$

$$M \sim |t|^{\beta}, \qquad \chi \equiv \frac{\partial M}{\partial H} \sim |t|^{-\gamma}$$

if $d<4$,

λ already here

$$\left[\mu \frac{d}{d\mu} + \sum_i \beta_i \frac{\partial}{\partial \rho_i} + 2\gamma\right] G(x; \mu, \{\rho_i\}) = 0$$

<u>Ansatz</u> : $G(x) = \frac{1}{|x|^{d-2}} g(\mu|x|, \{\rho_i\})$.

(Fourier transform of $G(p) \sim p^{-2}$)

Solution : $G(x) = \frac{1}{|x|^{d-2}} h(\{\rho_i(x)\}) \exp\left\{-2\int_{1/\mu}^{|x|} d\log|x'|\, \gamma(\{\bar{\rho}_i(x')\})\right\}$

and $\frac{d}{d\log(1/\mu|x|)} \bar{\rho}_i = \beta_i(\{\rho_i\})$,

$\bar{\rho}_m = \rho_m (\mu|x|)^{2-\gamma_{\phi^2}(\lambda_*)}$

Wilson-Fisher.

$\bar{\rho}_i = \rho_i (\mu|x|)^{-\Delta_i}$

at fixed point, $\rho_m \sim t$ (our choice).

$$G(x) = \frac{1}{|x|^{d-2}} \frac{1}{(\mu|x|)^{2\gamma(\lambda_*)}} h\left(t(\mu|x|)^{2-\gamma_{\phi^2}(\lambda_*)}\right) \quad (*)$$

from here $\underline{\underline{\eta = 2\gamma(\lambda_*)}}$.

$h \underset{|x|\to\infty}{\sim} \exp(-|x|(\mu t^{\nu}))$, because it's a scalar th.

see definition of $\xi$.

Comparing with $(*)$ we get $\nu = 1/(2-\gamma_{\varphi^2}(\lambda_*))$.

<u>NOW</u> : $\lambda_*^{W.F.} = \frac{16\pi^2}{3}(4-d)$, $\quad \gamma_{\varphi^2} = \frac{\lambda}{16\pi^2} = \frac{4-d}{3}$,

$$\nu = 3/(2+d).$$

However, this is a 1-loop result. One can improve it...

More generally, we expect any thermodynam. system with $N$ d.o.f. to be described by a CFT with $N$ scalar fields. (<u>Remark</u>: be careful with the perturbative statements in Peskin - there is some "cheating" there.)

<u>Example</u> : $O(N)$ symmetric $\phi^4$ theory :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{m^2}{2}(\varphi^i)^2 - \frac{\lambda}{4}(\phi^i\varphi^i)^2$$

in $d = 4-\varepsilon$.

$$\beta = -\epsilon\lambda + (N+8)\frac{\lambda^2}{8\pi^2}.$$

Wilson-Fisher point $\lambda_* = \frac{8\pi^2}{(N+8)\epsilon}$.

$$\gamma(\lambda_*) = \frac{N+2}{4(N+8)}\epsilon^2 + \dots, \quad \gamma_{\varphi^2}(\lambda_*) = \frac{N+2}{N+8}\epsilon + \dots$$

$$\nu^{-1} = 2 - \frac{N+2}{N+8}\epsilon + O(\epsilon^2)$$

For the magnetic (thermodynamic) properties we need an effective potential. To see the analogy, have a look to another lecture ("Convex and...") the purpose of the effective potential is to correct classical potential by quantum effects.

without proof:

$$\Gamma[\phi_{cl}] = i \sum_n \frac{1}{n!} \int dx_1 \dots dx_n \; \phi_{cl}(x_1) \dots \phi_{cl}(x_n) \, \Gamma^{(n)}(x_1, \dots x_n)$$

$\Gamma^{(n)}$ are 1PI amplitudes.

$$\Gamma^{(3)}(p_1, p_2, p_3) = \frac{G^{(3)}(p_1, p_2, p_3)}{G^{(2)}(p_1) \, G^{(2)}(p_2) \, G^{(2)}(p_3)}$$

So, we know how it rescales with the fields, i.e.

$\Gamma^{(n)} = Z(\mu)^{n/2} \, \Gamma_0^{(n)}$ and write the CS eqn:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - n \gamma(\lambda)\right] \Gamma^{(n)}(\{p_i\}; \mu, \lambda) = 0.$$

and

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma(\lambda) \int dx \, \phi_{cl}(x) \frac{\delta}{\delta \phi_{cl}(x)}\right] \Gamma(\phi_{cl}; \mu, \lambda) = 0$$

and, if $\phi_{cl} = \text{const}$, then

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-\Phi_{cl}\frac{\partial}{\partial\Phi_{cl}}\right]V_{eff}(\Phi_{cl},\mu,\lambda)=0$$

Now, let's do the replacement:

$$\Phi_{cl}\rightsquigarrow M,\quad J\rightsquigarrow H,\quad V_{eff}\rightsquigarrow G(M)$$

Gibbs energy

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i\beta_i\frac{\partial}{\partial\rho_i}-\gamma M\frac{\partial}{\partial M}\right]G(M,\mu,\{\rho_i\})=0$$

$\lambda$ already here

$[G]=\text{mass}^d$, $[\Phi]=\text{mass}^{(d-2)/2}$, thus

$$G(M,\mu,\{\rho_i\})=M^{\frac{2d}{d-2}}\,g\left((M\mu^{-\frac{d-2}{2}}),\{\rho_i\}\right).$$

Running coupling

$$\frac{d}{d\log M}\bar{\rho}_i=\frac{2\beta_i(\{\bar{\rho}_i\})}{d-2+2\gamma(\{\rho_i\})}$$

Again, we tune $\rho_m\sim t$ and $\bar{\rho}_i=0$ $(i\neq m)$, and pass close to the Wilson–Fisher point.

After all (see Peskin) we get the values of the critical exponents:

$$\beta = \frac{d-2+2\gamma(\lambda_*)}{2(2-\gamma_{\varphi^2}(\lambda_*))}$$

Scaling laws for $\varphi$ & $\varphi^2$ only!

$$\delta = \frac{2d}{d-2+2\gamma(\lambda_*)} - 1 = \frac{d+2-2\gamma(\lambda_*)}{d-2+2\gamma(\lambda_*)}$$

$$\alpha = 2\left(1 - \frac{d}{2-\gamma_{\varphi^2}}\right)$$

$$\gamma = \frac{2(1-\gamma(\lambda_*))}{2-\gamma_{\varphi^2}(\lambda_*)}$$

important:

$$\alpha = 2 - d\nu, \qquad \beta = \frac{\nu}{2}(d-2+\eta),$$

$$\gamma = (\delta-1)\beta \quad \ldots$$

Model-independent results!

Next page shows the comparison of the QFT results to an experiment.

**Table 13.1.** Values of Critical Exponents for Three-Dimensional Statistical Systems

| Exponent | Landau | QFT | Lattice | Experiment | |
|---|---|---|---|---|---|
| $N = 1$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.241 (2) | 1.239 (3) | 1.240 (7) | binary liquid |
| | | | | 1.22 (3) | liquid-gas |
| | | | | 1.24 (2) | $\beta$-brass |
| $\nu$ | 0.5 | 0.630 (2) | 0.631 (3) | 0.625 (5) | binary liquid |
| | | | | 0.65 (2) | $\beta$-brass |
| $\alpha$ | 0.0 | 0.110 (5) | 0.103 (6) | 0.113 (5) | binary liquid |
| | | | | 0.12 (2) | liquid-gas |
| $\beta$ | 0.5 | 0.325 (2) | 0.329 (9) | 0.325 (5) | binary liquid |
| | | | | 0.34 (1) | liquid-gas |
| $\eta$ | 0.0 | 0.032 (3) | 0.027(5) | 0.016 (7) | binary liquid |
| | | | | 0.04 (2) | $\beta$-brass |
| $N = 2$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.316 (3) | 1.32 (1) | | |
| $\nu$ | 0.5 | 0.670 (3) | 0.674 (6) | 0.672 (1) | superfluid $^4$He |
| $\alpha$ | 0.0 | −0.007 (6) | 0.01 (3) | −0.013 (3) | superfluid $^4$He |
| $N = 3$ Systems: | | | | | |
| $\gamma$ | 1.0 | 1.386 (4) | 1.40 (3) | 1.40 (3) | EuO, EuS |
| | | | | 1.33 (3) | Ni |
| | | | | 1.40 (3) | $RbMnF_3$ |
| $\nu$ | 0.5 | 0.705 (3) | 0.711 (8) | 0.70 (2) | EuO, EuS |
| | | | | 0.724 (8) | $RbMnF_3$ |
| $\alpha$ | 0.0 | −0.115 (9) | −0.09 (6) | −0.011 (2) | Ni |
| $\beta$ | 0.5 | 0.365 (3) | 0.37 (5) | 0.37 (2) | EuO, EuS |
| | | | | 0.348 (5) | Ni |
| | | | | 0.316 (8) | $RbMnF_3$ |
| $\eta$ | 0.0 | 0.033 (4) | 0.041 (14) | | |

The values of critical exponents in the column 'QFT' are obtained by resumming the perturbation series for anomalous dimensions at the Wilson-Fisher fixed point in $O(N)$-symmetric $\phi^4$ theory in three dimensions. The values in the column 'Lattice' are based on analysis of high-temperature series expansions for lattice statistical mechanical models. The values in the column 'Experiment' are taken from experiments on critical points in the systems described. In all cases, the numbers in parentheses are the standard errors in the last displayed digits. This table is based on J. C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* **B21**, 3976 (1980), with some values updated from J. Zinn-Justin (1993), Chapter 27. A full set of references for the last two columns can be found in these sources.