Zero curvature (Lax) representation and monodromy matrix.

Main idea: find a suitable representation of EOM, which, according to a standard recipe, will give us integrals of motion in involution.

One can consider an auxiliary problem (2D)

$$\begin{cases} \partial_x F = U(x,t,\lambda) F \\ \partial_t F = V(x,t,\lambda) F \end{cases} \quad \begin{matrix} \overleftarrow{\partial_t} \\ \overleftarrow{\partial_x} \end{matrix}$$

which can be resolved if (compatibility condition for the overdet. system)

$$\partial_t U - \partial_x V + [U, V] = 0, \qquad \text{i.e.}$$

$$[\partial_t - V, \ \partial_x - U] = 0 \quad - \text{ zero curvature condition}$$

like $[D_\mu \, D_\nu] \sim F_{\mu\nu}$ with V and U playing a role of connection components. (Lax pair)

Example 1 Sine - Gordon model

$$\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 + \frac{4m^2}{\beta^2}(\cos 2\beta\varphi - 1)$$

EOM: $\partial_t^2 \varphi - \partial_x^2 \varphi = -8 m^2 \beta^{-1} \sin 2\beta\varphi$

now, one can show that the pair (U, V) is

$$U(x,t,\lambda) = \begin{pmatrix} i\frac{\beta}{2}\dot{\varphi} & im\left(\lambda e^{-i\beta\varphi} - \frac{1}{\lambda}e^{i\beta\varphi}\right) \\ im\left(\lambda e^{i\beta\varphi} - \frac{1}{\lambda}e^{-i\beta\varphi}\right) & -i\frac{\beta}{2}\dot{\varphi} \end{pmatrix}$$

$$V(x,t,\lambda) = \begin{pmatrix} i\frac{\beta}{2}\varphi' & im\left(\lambda e^{-i\beta\varphi} + \frac{1}{\lambda}e^{i\beta\varphi}\right) \\ im\left(\lambda e^{i\beta\varphi} + \frac{1}{\lambda}e^{-i\beta\varphi}\right) & -i\frac{\beta}{2}\varphi' \end{pmatrix}$$

in other words,

$$U = i\frac{\beta}{2}\dot{\varphi}\,\sigma_3 + k_0 \cdot \sin\beta\varphi \cdot \sigma_2 + k_1 \cos\beta\varphi \cdot \sigma_1$$

$$V = i\frac{\beta}{2}\varphi'\,\sigma_3 + k_1 \sin\beta\varphi \cdot \sigma_2 + k_0 \cos\beta\varphi \cdot \sigma_1,$$

where $k_0 \equiv im\left(\lambda + \frac{1}{\lambda}\right)$, $\quad k_1 \equiv im\left(\lambda - \frac{1}{\lambda}\right)$

<u>Comments</u>:

1) light cone rep: $[\partial_+ - U_+,\ \partial_- - U_-] = 0,$

where $U_\pm \equiv \frac{1}{2}(U \pm V)$

2) gauge transformation: $[\partial_+ - \tilde{U}_+,\ \partial_- - \tilde{U}_-] = 0,$

where

$$\partial_+ - \tilde{U}_+ = g^{-1}(\partial_+ - U_+)g$$

$$\partial_- - \tilde{U}_- = g^{-1}(\partial_- - U_-)g, \qquad g = \begin{pmatrix} e^{i\frac{\beta}{2}\tilde{\varphi}} & 0 \\ 0 & e^{-i\frac{\beta}{2}\tilde{\varphi}} \end{pmatrix} = \exp\left\{\frac{i\beta\tilde{\varphi}\sigma_3}{2}\right\}.$$

Important statement: if we're able to write down the zero curvature condition, then the system is integrable. By "integrable" we mean that there are enough integrals of motion in the involution. (infinite number of them for solitons).

Let's consider a problem with periodic bndry conditions:

$$U(x+2L, t, \lambda) = U(x, t, \lambda)$$
$$V(x+2L, t, \lambda) = V(x, t, \lambda)$$

then the monodromy matrix T_L is the matrix of parallel transport along x- direction:

$$T_L(\lambda, t_0) = \mathbb{P}\exp\left\{\int_{-L}^{L} U(x, t_0, \lambda)\, dx\right\}$$

compare to Wilson line.

it has an interesting property:

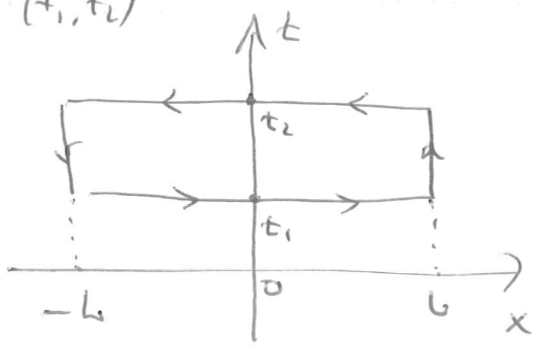
According to (ZCC) $\quad \Omega_\gamma \equiv \mathbb{P}\exp\left(\int_\gamma U dx + V dt\right) = \mathbb{1}$

along a closed contour (compare to Wilson loop).

then $S_-^{-1}\, T_L^{-1}(t_2)\, S_+\, T_L(t_1) = \mathbb{1}$,

where $S_\pm(\lambda, t_1, t_2) = \mathbb{P}\exp\int_{t_1}^{t_2} V(\pm L, t, \lambda)\, dt$.

: So, $T_L(t_2) = S_+(t_1,t_2)\, T_L(t_1)\, S_+^{-1}(t_1,t_2)$



Due to the periodicity

$$S_+ = S_-$$

and therefore

$$\mathrm{tr}\, T_L(\lambda, t_2) = \mathrm{tr}\, T_L(\lambda, t_1), \quad \text{where}$$

trace is the ~~trace~~ matrix trace in $\mathbb{C}^2$, so we conclude that it doesn't depend on t, and $F_L(\lambda) = \mathrm{tr}\, T_L(\lambda)$ is a generating function for the conservation laws.

Properties of the monodromy matrix:

1) it satisfies the dif. eqn. of the auxiliary linear problem: (generalization: $T(\lambda) \to T(x,y,\lambda)$.

$$\frac{\partial}{\partial x} T(x,y,\lambda) = U(x,\lambda)\, T(x,y,\lambda) \quad \text{with the}$$

initial condition $T(x,y,\lambda)\big|_{x=y} = \mathbb{1}$

2) Superposition property: $T(x,z,\lambda)\, T(z,y,\lambda) = T(x,y,\lambda)$

3) $T(x,y,y)$ is unimodular, $\det T(x,y,\lambda) = 1$,

since $\mathrm{tr}\, U(x,\lambda) = 0$ (σ- matrices). in Sine-G.

Example 2 Toda lattice

$$H = \sum_{n=1}^{N} \left(\frac{p_n^2}{2} + e^{q_{n-1} - q_n} \right), \qquad q_{N+1} \equiv q_1$$

$$\dot{q}_n = p_n, \qquad \dot{p}_n = -\frac{\partial H}{\partial q_n} = e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}$$

(ZCC): $\dot{L}_n = M_{n+1} L_n - L_n M_n$, where

$$L_n = \begin{pmatrix} 0 & e^{q_n} \\ -e^{-q_n} & \lambda - p_n \end{pmatrix}, \qquad M_{n+1} = \begin{pmatrix} 0 & -e^{q_n} \\ e^{-q_{n+1}} & -\lambda \end{pmatrix}$$

So, use the method of the monodromy matrix:

$$T(\lambda) \equiv \mathrm{tr}\left[L_N(\lambda) L_{N-1}(\lambda) \ldots L_1(\lambda) \right]$$

(parallel transport around the lattice), $\dot{T}(\lambda) = 0$

$$T(\lambda) = \lambda^N - P\lambda^{N-1} + \left(\frac{1}{2} P^2 - H \right) \lambda^{N-2} + \ldots =$$

$$= \sum_{k=0}^{N} \lambda^k H_k$$ integrals of motion!

to be more precise,

$$\sum_{\substack{n,m \\ n \neq m}} p_n p_m = \sum_{n,m} p_n p_m - \sum p_n^2 = P^2 - 2H$$

H_K are functionally independent (in involution + rank $\left(\frac{\partial H_m}{\partial p_n}, \frac{\partial H_m}{\partial q_n} \right) = N$)

- There is also an alternative formulation of this recipe:

Suppose we've found a pair of $n \times n$ matrices (L, M):

$$\dot{L} = [M, L] \quad \text{equivalent to EOM.}$$

then the time dependence can be described by

$$L(t) = U(t) L_0 U^{-1}(t), \qquad M = \dot{U}(t) U^{-1}(t)$$

then $$\dot{I}_k = \operatorname{tr} L^{k-1} [L, M] = 0,$$

where $$I_k = \frac{1}{k} \operatorname{tr} L^k, \qquad k = \overline{1, n}$$

and they are integrals of motion.

if I_k are in involution and $n \geq \#$ d.o.f.,

then the system is integrable.

for system with # d.o.f. $\simeq \infty$ it's difficult, so we introduce isospectral deformation which ensures the existence of sufficiently many integrals of motion

$$I_k(\lambda) = \sum_m I_{k,m} \lambda^m,$$

their involutivity follows from the Yang-Baxter equation.